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WEITZENBÖCK DERIVATIONS AND CLASSICAL INVARIANT THEORY: I. POINCARÉ SERIES

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Abstract. By using classical invariant theory approach, formulas for computation of the Poincaré series of the kernel of linear locally nilpotent derivations are found.

1. Introduction. Let $\mathbb{K}$ be a field of characteristic 0. A derivation $\mathcal{D}$ of the polynomial algebra $\mathbb{K}[\mathcal{Z}_n]$, $\mathcal{Z}_n=\{z_1,z_2,\ldots,z_n\}$ is called a linear derivation if

$$\mathcal{D}(z_i)=\sum_{j=1}^{n} a_{i,j}z_j, a_{i,j}\in\mathbb{K},\quad i=1,\ldots,n.$$

A linear derivation $\mathcal{D}$ is called a Weitzenböck derivation if the matrix $A_{\mathcal{D}}:=\{a_{i,j}\}_{i,j=1}^n$ is nilpotent. It is clear that a Weitzenböck derivation is a locally nilpotent derivation of $\mathbb{K}[z_1,z_2,\ldots,z_n]$. Any Weitzenböck derivation $\mathcal{D}$ is completely determined by the Jordan normal form of the matrix $A_{\mathcal{D}}$. We will

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denote by $\mathcal{D}_{\mathbf{d}}$, $\mathbf{d} := (d_1, d_2, \ldots, d_s)$ the Weitzenbök derivation with the Jordan normal form of the matrix $A_{\mathcal{D}_{\mathbf{d}}}$consisting of s Jordan blocks of size d_1+1, d_2+1, $\ldots, d_s+1$, respectively. The only derivation which corresponds to a single Jordan block of size $d+1$ is called a basic Weitzenböck derivation and denoted by Δ_d.

The algebra

$$\ker D_{\mathbf{d}} = \{f \in \mathbb{K}[\mathcal{Z}_n] \mid \mathcal{D}_{\mathbf{d}}(f) = 0\}$$

is called the kernel of the derivation $D_{\mathbf{d}}$. It is well known that the kernel $\ker \mathcal{D}_{\mathbf{d}}$ is a finitely generated algebra, see [20]–[19]. However, it remains an open problem to find a minimal system of homogeneous generators (or even the cardinality of such a system) of the algebra $\ker \mathrm{D}_{\mathbf{d}}$ even for small tuples $\mathbf{d}$.

On the other hand, the problem to describe the kernel $\ker \mathcal{D}_{\mathbf{d}}$ can be reduced to an old problem of classical invariant theory, namely to the problem to describe the algebra of joint covariants of several binary forms. In fact, it is well known that there is a one-to-one correspondence between the $\mathbb{G}_a$-actions on an affine algebraic variety V and the locally nilpotent $\mathbb{K}$-derivations on its algebra of polynomial functions. Let us identify the algebra $\mathbb{K}[\mathcal{Z}_n]$ with the algebra $\mathcal{O}[\mathbb{K}^n]$ of polynomial functions of the algebraic variety $\mathbb{K}^n$. Then, the kernel of the derivation $\mathcal{D}_{\mathbf{d}}$ coincides with the invariant ring of the induced via $\exp(t\,\mathcal{D}_{\mathbf{d}})$ action:

$$\ker \mathcal{D}_{\mathbf{d}} = \mathbb{K}[\mathcal{Z}_n]^{\mathbb{G}_a} \cong \mathcal{O}(\mathbb{K}^n)^{\mathbb{G}_a}.$$

Now, let $B_{d_1}, B_{d_2}, \ldots B_{d_s}$ be the vector $\mathbb{K}$-spaces of binary forms of degrees $d_1, d_2, \ldots, d_s$ endowed with the natural action of the group SL_2. Consider the induced action of the group SL_2 on the algebra of polynomial functions $\mathcal{O}[B_{\mathbf{d}} \oplus \mathbb{K}^2]$ on the vector space $B_{\mathbf{d}} \oplus \mathbb{K}^2$, where

$$B_{\mathbf{d}} := B_{d_1} \oplus B_{d_2} \oplus \cdots \oplus B_{d_s}, \quad \dim(B_{\mathbf{d}}) = d_1 + d_2 + \cdots + d_s + s.$$

Let

$$U_2 = \left\{ \left(\begin{array}{cc} 1 & \lambda \\ 0 & 1 \end{array} \right) \Big| \lambda \in \mathbb{K} \right\}$$

be the maximal unipotent subgroup of the group SL_2. The application of the Grosshans principle, see [10], [14], gives

$$\mathcal{O}[B_{\mathbf{d}} \oplus \mathbb{K}^2]^{SL_2} \cong \mathcal{O}[B_{\mathbf{d}}]^{U_2}.$$

Thus

$$\mathcal{O}[B_{\mathbf{d}} \oplus \mathbb{K}^2]^{\mathfrak{sl}_2} \cong \mathcal{O}[B_{\mathbf{d}}]^{\mathfrak{u}_2}.$$

Since $U_2 \cong (\mathbb{K}, +)$ and $\mathbb{K}z_1 \oplus \mathbb{K}z_2 \oplus \cdots \oplus \mathbb{K}z_n \cong B_{\mathbf{d}}$ it follows

$$\ker \mathcal{D}_{\mathbf{d}} \cong \mathcal{O}[B_{\mathbf{d}} \oplus \mathbb{K}^2]^{\mathfrak{sl}_2}.$$

In the language of classical invariant theory the algebra $\mathcal{C}_{\mathbf{d}} := \mathcal{O}[B_{\mathbf{d}} \oplus \mathbb{K}^2]^{\mathfrak{sl}_2}$ is called the algebra of joint covariants of s binary forms, the algebra $\mathcal{S}_{\mathbf{d}} := \mathcal{O}[B_{\mathbf{d}}]^{\mathfrak{u}_1}$ is called the algebra of joint semi-invariants of binary forms and the algebra $\mathcal{I}_{\mathbf{d}} := \mathcal{O}[B_{\mathbf{d}}]^{\mathfrak{sl}_2}$ is called the algebra of invariants of the s binary forms of degrees $d_1, d_2, \ldots, d_s$. Algebras of joint covariants of binary forms were an object of research in the invariant theory in the 19th century.

The reductivity of SL_2 implies that the algebras $\mathcal{I}_{\mathbf{d}}$, $\mathcal{S}_{\mathbf{d}} \cong \ker \mathcal{D}_{\mathbf{d}}$ are finitely generated $\mathbb{N}$-graded algebras. The formal power series $\mathcal{PI}_{\mathbf{d}}, \mathcal{PD}_{\mathbf{d}} = \mathcal{PS}_{\mathbf{d}} \in \mathbb{Z}[[z]]$,

$$\mathcal{PI}_{\mathbf{d}}(z) = \sum_{i=0}^{\infty} \dim((\mathcal{I}_{\mathbf{d}})_i) z^i, \mathcal{PS}_{\mathbf{d}}(z) = \sum_{i=0}^{\infty} \dim((\mathcal{S}_{\mathbf{d}})_i) z^i,$$

are called the Poincaré series of the algebras of joint invariants and semi-invariants. The finitely generation of the algebras $\mathcal{I}_{\mathbf{d}}$ and $\mathcal{S}_{\mathbf{d}}$ implies that their Poincaré series are the power series expansions of certain rational functions. We consider here the problem of computing efficiently these rational functions. It could be the first step towards describing these algebras.

Let us recall that the Poincaré series of the algebra of covariants of a binary d-form equals the Poincaré series of the kernel of the basic Weitzenböck derivation Δ_d. For the cases $d \leq 10$, $d = 12$ the Poincaré series of the algebra of invariants and covariants for the binary d-form were calculated by Sylvester and Franklin, see [17], [18]. For the purpose, they used the Cayley-Sylvester formula for the dimension of graded subspaces. In [13] the Poincaré series for Δ_5 was rediscovered. Springer [16] derived a formula for computing the Poincaré series of the algebras of invariants of the binary d-forms. This formula has been used by Brouwer and Cohen [4] for the Poincaré series calculations in the cases $d \leq 17$ and also by Littelmann and Procesi [12] for even $d \leq 36$. For the case $d \leq 30$ in [3] the explicit form of the Poincare series is given.

In [1], [2] we have found Cayley-Sylvester type and Springer type formulas for the basic derivation Δ_d and for the derivation $\mathcal{D}_{\mathbf{d}}$ for $\mathbf{d} = (d_1, d_2)$. Also, for those derivations the Poincaré series was found for $d, d_1, d_2 \leq 30$. Relatively recently, in [7] a formula for computing the Poincaré series of the Weitzenböck derivation $\mathcal{D}_{\mathbf{d}}$ for arbitrary $\mathbf{d}$ was announced.

In this paper we prove Cayley-Sylvester type formulas for calculation of $\dim(\mathcal{I}_{\mathbf{d}})_i$, $\dim(\ker \mathcal{D}_{\mathbf{d}})_i$ and Springer-type formulas for calculation of $\mathcal{PI}_{\mathbf{d}}(z)$,

$\mathcal{PS}_{\mathbf{d}}(z) = \mathcal{PD}_{\mathbf{d}}(z)$ for arbitrary $\mathbf{d}$. Also, for the cases $\mathbf{d} = (1,1,\ldots,1)$, $\mathbf{d} = (2,2,\ldots,2)$ the explicit formulas for $\mathcal{PD}_{\mathbf{d}}(z)$ are given.

2. Cayley-Sylvester type formula for the kernel. To begin, we give a proof of a Cayley-Sylvester type formula for the dimension of graded subspaces of the kernel of Weitzenböck derivation $\mathcal{D}_{\mathbf{d}}$, $\mathbf{d} := (d_1, d_2, \ldots, d_s)$.

Let us consider the polynomial algebra $\mathbb{K}[X_{\mathbf{d}}]$ in the set of variables

$$X_{\mathbf{d}} := \left\{x_0^{(1)}, x_1^{(1)}, \ldots, x_{d_1}^{(1)}, x_0^{(2)}, x_1^{(2)}, \ldots, x_{d_2}^{(2)}, \ldots x_0^{(s)}, x_1^{(s)}, \ldots, x_{d_s}^{(s)}\right\}.$$

Define on $\mathbb{K}[X_{\mathbf{d}}]$ the Weitzenböck derivation $\mathcal{D}_{\mathbf{d}}$, $\mathbf{d} := (d_1, d_2, \ldots, d_s)$ by

$$\mathcal{D}_{\mathbf{d}}(x_i^{(k)}) = i\, x_{i-1}^{(k)}, i = 0, \ldots, d_k, k = 1, \ldots, s.$$

Also, define on $\mathbb{K}[X_{\mathbf{d}}]$ the following linear derivations $\mathcal{D}_{\mathbf{d}}^*$ and $\mathcal{E}_{\mathbf{d}}$, by

$$\mathcal{D}_{\mathbf{d}}^*(x_i^{(k)}) = (d_k - i)\, x_{i+1}^{(k)}, \mathcal{E}_{\mathbf{d}}(x_i^{(k)}) = (d_k - 2\,i)\, x_i^{(k)}, k = 1, \ldots, s.$$

By direct calculation we get

$$[\mathcal{D}_{\mathbf{d}}, \mathcal{D}_{\mathbf{d}}^*]\,(x_i^{(k)}) = \mathcal{D}_{\mathbf{d}}\left(\mathcal{D}_{\mathbf{d}}^*\left(x_i^{(k)}\right)\right) - \mathcal{D}_{\mathbf{d}}^*\left(\mathcal{D}_{\mathbf{d}}\left(x_i^{(k)}\right)\right) = (d_k - 2i)\, x_i^{(k)} = \mathcal{E}_{\mathbf{d}}(x_i^{(k)}).$$

In the same way we get $[\mathcal{D}_{\mathbf{d}}, \mathcal{E}_{\mathbf{d}}] = -2\mathcal{D}_{\mathbf{d}}$ and $[\mathcal{D}_{\mathbf{d}}^*, \mathcal{E}_{\mathbf{d}}] = 2\mathcal{D}_{\mathbf{d}}^*$. Therefore, the polynomial algebra $\mathbb{K}[X_{\mathbf{d}}]$ considered as a vector space becomes an $\mathfrak{sl}_2$-module.

Let $\mathfrak{u}_2 = \mathbb{K}[X_{\mathbf{d}}]$ be the maximal unipotent subalgebra of $\mathfrak{sl}_2$. Let us identify the following algebras

$$\mathbb{K}[X_{\mathbf{d}}]^{\mathfrak{sl}_2} = \{v \in \mathbb{K}[X_{\mathbf{d}}] \mid \mathcal{D}_{\mathbf{d}}(v) = \mathcal{D}_{\mathbf{d}}^*(v) = 0\},$$
$$\ker \mathcal{D}_{\mathbf{d}} = \mathbb{K}[X_{\mathbf{d}}]^{\mathfrak{u}_2} = \{v \in \mathbb{K}[X_{\mathbf{d}}] \mid \mathcal{D}_{\mathbf{d}}(v) = 0\},$$

with the algebra of joint invariants $\mathcal{I}_{\mathbf{d}}$ and the algebra of joint semi-invariant $\mathcal{S}_{\mathbf{d}}$ of the binary forms of the degrees $\mathbf{d} = (d_1, d_2, \ldots, d_s)$, respectively. For any element $v \in \mathcal{S}_{\mathbf{d}}$ the natural number r is called the order of the element v if r is the smallest natural number such that

$$(\mathcal{D}_{\mathbf{d}}^*)^r(v) \neq 0, (\mathcal{D}_{\mathbf{d}}^*)^{r+1}(v) = 0.$$

It is clear that any semi-invariant of order r is the highest weight vector for an irreducible $\mathfrak{sl}_2$-module of dimension $r+1$ in $\mathbb{K}[X_{\mathbf{d}}]$.

The algebra of simultaneous covariants $\mathcal{C}_{\mathbf{d}}$ is isomorphic to the algebra of simultaneous semi-invariants. Therefore, it is enough to compute the Poincaré series of the algebra $\mathcal{S}_{\mathbf{d}} \cong \ker \mathcal{D}_{\mathbf{d}}$.

The algebras $\mathbb{K}[X_{\mathbf{d}}]$, $\mathcal{I}_{\mathbf{d}}$, $\mathcal{S}_{\mathbf{d}}$ are graded algebras:

$$\begin{aligned}
&\mathbb{K}[X_{\mathbf{d}}] = (\mathbb{K}[X_{\mathbf{d}}])_0 + (\mathbb{K}[X_{\mathbf{d}}])_1 + \cdots + (\mathbb{K}[X_{\mathbf{d}}])_m + \cdots ,\\
&\mathcal{I}_{\mathbf{d}} = (\mathcal{I}_{\mathbf{d}})_0 + (\mathcal{I}_{\mathbf{d}})_1 + \cdots + (\mathcal{I}_{\mathbf{d}})_m + \cdots ,\\
&\mathcal{S}_{\mathbf{d}} = (\mathcal{S}_{\mathbf{d}})_0 + (\mathcal{S}_{\mathbf{d}})_1 + \cdots + (\mathcal{S}_{\mathbf{d}})_m + \cdots .
\end{aligned}$$

and each $(\mathbb{K}[X_{\mathbf{d}}])_m$ is a completely reducible representation of the Lie algebra $\mathfrak{sl}_2$.

Let V_k be the standard irreducible $\mathfrak{sl}_2$-module, $\dim V_k = k+1$. Then, the following primary decomposition holds

$$(1) \qquad (\mathbb{K}[X_{\mathbf{d}}])_m \cong \gamma_m(\mathbf{d};0)V_0 + \gamma_m(\mathbf{d};1)V_1 + \cdots + \gamma_m(\mathbf{d};m\cdot d^*)V_{m\,d^*},$$

here $d^* := \max(d_1, d_2, \ldots d_s)$ and $\gamma_m(\mathbf{d};k)$ is the multiplicity of the representation V_k in the decomposition of $(\mathbb{K}[X_{\mathbf{d}}])_m$. On the other hand, the multiplicity $\gamma_m(\mathbf{d};k)$ is equal to the number of linearly independent homogeneous simultaneous semi-invariants of degree m and order k. In particular, the number of linearly independent simultaneous invariants of degree m is equal to $\gamma_m(\mathbf{d};0)$. These arguments prove

Lemma 2.1.

(*i*) $\dim(\mathcal{I}_{\mathbf{d}})_m = \gamma_m(\mathbf{d};0)$;

(*ii*) $\dim(S_{\mathbf{d}})_m = \gamma_m(\mathbf{d};0) + \gamma_m(\mathbf{d};1) + \cdots + \gamma_m(\mathbf{d};m\,d^*)$.

Let us recall some general facts about the representations of the Lie algebra $\mathfrak{sl}_2$.

Denote by Λ_W the set of weights of the representation W, for instance

$$\Lambda_{V_d} = \{-d, -d+2, \ldots, d\}.$$

The formal sum

$$\mathrm{Char}(W) = \sum_{\lambda \in \Lambda_W} n_W(\lambda) q^{\lambda},$$

is called the character of the representation W, here $n_W(\lambda)$ denotes the multiplicity of the weight $\lambda \in \Lambda_W$. Since the multiplicity of any weight of the irreducible representation V_d is equal to 1, we have

$$\mathrm{Char}(V_d) = q^{-d} + q^{-d+2} + \cdots + q^{d}.$$

Consider the set of variables: $x_0^{(1)}, x_1^{(1)}, \ldots, x_{d_1}^{(1)}$, $x_0^{(2)}, x_1^{(2)}, \ldots, x_{d_2}^{(2)}$, …, $x_0^{(s)}$, $x_1^{(s)}$, …, $x_{d_s}^{(s)}$. The character $\mathrm{Char}\,((\mathbb{K}[X_{\mathbf{d}}])_m)$ of the representation $(\mathbb{K}[X_{\mathbf{d}}])_m$, see [8], equals

$$H_m(q^{-d_1}, q^{-d_1+2}, \ldots, q^{d_1}, q^{-d_2}, q^{-d_2+2}, \ldots, q^{d_2}, \ldots, q^{-d_s}, q^{-d_s+2}, \ldots, q^{d_s}),$$

where $H_m(x_0^{(1)}, x_1^{(1)}, \ldots, x_{d_1}^{(1)}, \ldots, x_0^{(s)}, x_1^{(s)}, \ldots, x_{d_s}^{(s)})$ is the complete symmetrical function

$$H_m(x_0^{(1)}, x_1^{(1)}, \ldots, x_{d_1}^{(1)}, \ldots, x_0^{(s)}, x_1^{(s)}, \ldots, x_{d_s}^{(s)}) =$$

$$= \sum_{|\alpha^{(1)}|+\cdots+|\alpha^{(s)}|=m} (x_0^{(1)})^{\alpha_0^{(1)}} (x_1^{(1)})^{\alpha_1^{(1)}} \ldots (x_{d_1}^{(1)})^{\alpha_{d_1}^{(1)}} \cdots (x_0^{(s)})^{\alpha_0^{(s)}} (x_1^{(s)})^{\alpha_1^{(s)}} \ldots (x_{d_s}^{(1)})^{\alpha_{d_s}^{(s)}},$$

and $|\alpha^{(k)}| := \sum_{i=0}^{d_i} \alpha_i^{(k)}$.

By replacing $x_i^{(k)} = q^{d_k - 2\,i}$, we obtain the specialized expression for the character $(\mathbb{K}[X_{\mathbf{d}}])_m$, namely

$$\mathrm{Char}((\mathbb{K}[X_{\mathbf{d}}])_m) =$$

$$= \sum_{|\alpha^{(1)}|+\cdots+|\alpha^{(s)}|=n} (q^{d_1})^{\alpha_0^{(1)}} (q^{d_1-2\cdot 1})^{\alpha_1^{(1)}} \ldots (q^{-d_1})^{\alpha_{d_1}^{(1)}} \ldots (q^{d_s})^{\alpha_0^{(s)}} (q^{d_s-2\cdot 1})^{\alpha_1^{(s)}} \ldots (q^{-d_s})^{\alpha_{d_s}^{(s)}} =$$

$$= \sum_{|\alpha^{(1)}|+\cdots+|\alpha^{(s)}|=n} q^{d_1|\alpha^{(1)}|+\cdots+d_s|\alpha^{(s)}| - 2\left(\alpha_1^{(1)}+2\alpha_2^{(1)}+\cdots+d_1\,\alpha_{d_1}^{(1)}\right) - \cdots - 2\left(\alpha_1^{(s)}+2\alpha_2^{(s)}+\cdots+d_s\,\alpha_{d_s}^{(s)}\right)} =$$

$$= \sum_{i=-m\,d^*}^{m\,d^*} \omega_n(\mathbf{d}; i) q^i,$$

here $\omega_m(\mathbf{d}; i)$ is the number of non-negative integer solutions of the following system of equations:

$$(2) \qquad \begin{cases} d_1|\alpha^{(1)}| + \cdots + d_s|\alpha^{(s)}| - 2\left(\alpha_1^{(1)} + 2\alpha_2^{(1)} + \cdots + d_1\,\alpha_{d_1}^{(1)}\right) - \\ - \cdots - 2\left(\alpha_1^{(s)} + 2\alpha_2^{(s)} + \cdots + d_s\,\alpha_{d_s}^{(s)}\right) = i \\ \\ |\alpha^{(1)}| + \cdots + |\alpha^{(s)}| = m. \end{cases}$$

We can summarize what we have shown so far in

Theorem 2.1.

$$\begin{aligned} &(i) \quad \dim(\mathcal{I}_{\mathbf{d}})_m = \omega_m(\mathbf{d};0) - \omega_m(\mathbf{d};2),\\ &(ii) \quad \dim(\mathcal{S}_{\mathbf{d}})_m = \omega_m(\mathbf{d};0) + \omega_m(\mathbf{d};1).\end{aligned}$$

P r o o f. (i) The zero weight appears once in any representation V_k, for even k, therefore

$$\omega_m(\mathbf{d};0) = \gamma_m(\mathbf{d};0) + \gamma_m(\mathbf{d};2) + \gamma_m(\mathbf{d};4) + \cdots$$

The weight 2 appears once in any representation V_k, for even $k > 0$, therefore

$$\omega_m(\mathbf{d};2) = \gamma_m(\mathbf{d};2) + \gamma_m(\mathbf{d};4) + \gamma_m(\mathbf{d};6) + \cdots$$

Taking into account Lemma 2.1, we obtain

$$\omega_m(\mathbf{d};0) - \omega_m(\mathbf{d};2) = \gamma_m(\mathbf{d};0) = \dim(\mathcal{I}_{\mathbf{d}})_m.$$

(ii) The weight 1 appears once in any representation V_k, for odd k, therefore

$$\omega_m(\mathbf{d};1) = \gamma_m(\mathbf{d};1) + \gamma_m(\mathbf{d};3) + \gamma_m(\mathbf{d};5) + \cdots$$

Thus,

$$\begin{aligned} &\omega_m(\mathbf{d};0) + \omega_m(\mathbf{d};1) =\\ &= \gamma_m(\mathbf{d};0) + \gamma_m(\mathbf{d};1) + \gamma_m(\mathbf{d};2) + \cdots + \gamma_m(\mathbf{d};n\,d^*) =\\ &= \dim(S_{\mathbf{d}})_m. \end{aligned}$$

□

Simplify the system (2) to

$$\begin{cases} d_1\alpha_0^{(1)} + (d_1-2)\alpha_1^{(1)} + (d_1-4)\alpha_2^{(1)} + \cdots + (-d_1)\,\alpha_{d_1}^{(1)} + \cdots + \\ +d_s\alpha_0^{(s)} + (d_s-2)\alpha_1^{(s)} + (d_s-4)\alpha_2^{(s)} + \cdots + (-d_s)\,\alpha_{d_s}^{(s)} = i,\\ \alpha_0^{(1)} + \alpha_1^{(1)} + \cdots + \alpha_{d_1}^{(1)} + \cdots + \alpha_0^{(s)} + \alpha_1^{(s)} + \cdots + \alpha_{d_s}^{(s)} = n. \end{cases}$$

It is well-known that the number $\omega_m(\mathbf{d};i)$ of non-negative integer solutions of the above system is equal to the coefficient of $t^m z^i$ of the generating function

$$f_{\mathbf{d}}(t,z) =$$

$$= \frac{1}{(1-tz^{d_1})(1-t\,z^{d_1-2})\cdots(1-t\,z^{-d_1})\cdots(1-tz^{d_s})(1-t\,z^{d_s-2})\cdots(1-t\,z^{-d_s})}.$$

Denote it in such a way: $\omega_m(\mathbf{d};i) := \left[t^m z^i\right](f_{\mathbf{d}}(t,z))$. Observe that $f_{\mathbf{d}}(t,z) = f_{\mathbf{d}}(t,z^{-1})$.

The following statement holds

Theorem 2.2.

$$\begin{array}{ll} (i) & \dim(I_{\mathbf{d}})_m = [t^m](1-z^2)f_{\mathbf{d}}(t,z), \\ (ii) & \dim(\mathcal{S}_{\mathbf{d}})_m = [t^m](1+z)f_{\mathbf{d}}(t,z). \end{array}$$

P r o o f. Taking into account the formal property $[x^{i-k}]f(x) = [x^i](x^k f(x))$, we get

$$\begin{aligned} &\dim(I_{\mathbf{d}})_m = \omega_m(\mathbf{d};0) - \omega_m(\mathbf{d};2) = [t^m]f_{\mathbf{d}}(t,z) - [t^m\,z^2]f_{\mathbf{d}}(t,z) = \\ &= [t^m]f_{\mathbf{d}}(t,z) - [t^m]z^{-2}f_{\mathbf{d}}(t,z) = [t^m]f_{\mathbf{d}}(t,z) - [t^m]z^2 f_{\mathbf{d}}(t,z^{-1}) = \\ &= [t^m](1-z^2)f_{\mathbf{d}}(t,z). \end{aligned}$$

In the same way

$$\begin{aligned} &\dim(S_{\mathbf{d}})_m = \omega_m(\mathbf{d};0) + \omega_m(\mathbf{d};1) = [t^m]f_{\mathbf{d}}(t,z) + [t^m\,z]f_{\mathbf{d}}(t,z) = \\ &= [t^m]f_{\mathbf{d}}(t,z) + [t^m]z^{-1}f_{\mathbf{d}}(t,z) = [t^m](1+z)f_{\mathbf{d}}(t,z). \qquad \square \end{aligned}$$

It is easy to see that the dimensions $\dim(I_{\mathbf{d}})_m$ and $\dim(S_{\mathbf{d}})_m$ allow the following representations:

$$\dim(I_{\mathbf{d}})_m = [t^m]\frac{1}{2\pi i}\oint_{|z|=1}(1-z^2)f_{\mathbf{d}}(t,z)\frac{dz}{z},$$

$$\dim(\mathcal{S}_{\mathbf{d}})_m = [t^m]\frac{1}{2\pi i}\oint_{|z|=1}(1+z)f_{\mathbf{d}}(t,z)\frac{dz}{z}.$$

3. Springer type formulas for the Poincaré series. Let us prove Springer type formulas for the Poincaré series $\mathcal{PI}_{\mathbf{d}}(z)$, $\mathcal{PS}_{\mathbf{d}}(z) = \mathcal{PD}_{\mathbf{d}}(z)$.

Consider the $\mathbb{C}$-algebra $\mathbb{C}[[t,z]]$ of a formal power series. For an arbitrary $m,n \in \mathbb{Z}^+$ define $\mathbb{C}$-linear function

$$\Psi_{m,n} : \mathbb{C}[[t,z]] \to \mathbb{C}[[z]],$$

in the following way:

$$\Psi_{m,n}\left(\sum_{i,j=0}^{\infty} a_{i,j}\, t^i z^j\right) = \sum_{i=0}^{\infty} a_{im,in} z^i.$$

Denote by φ_n the restriction of $\Psi_{m,n}$ to $\mathbb{C}[[z]]$, namely

$$\varphi_n\left(\sum_{i=0}^{\infty} a_i z^i\right) = \sum_{i=0}^{\infty} a_{in} z^i.$$

There is an effective algorithm of calculation for the function φ_n, see [1]. In some cases calculation of the functions Ψ can be reduced to calculation of the functions φ. The following statements hold:

Lemma 3.1. *For $R(z) \in \mathbb{C}[[z]]$ and for $m, n, k \in \mathbb{N}$ we have:*

$$\Psi_{1,n}\left(\frac{R(z)}{(1-tz^k)^m}\right) = \begin{cases} \dfrac{1}{(m-1)!}\dfrac{d^{m-1}(z^{m-1}\,\varphi_{n-k}(R(z)))}{dz^{m-1}}, & \text{if } n > k; \\[2ex] \dfrac{R(0)}{(1-z)^m}, & \text{if } n = k; \\[2ex] R(0), & \text{if } k > n. \end{cases}$$

P r o o f. Let $R(z) = \sum_{j=0}^{\infty} r_j z^j$. Observe, that

$$\frac{1}{(1-x)^m} = \frac{1}{(m-1)!}\frac{d^{m-1}}{dx^{m-1}}\left(\frac{1}{1-x}\right) = \sum_{i=0}^{\infty}\binom{s+m-1}{m-1}x^s.$$

Then for $n > k$ we have

$$\Psi_{1,n}\left(\frac{R(z)}{(1-tz^k)^m}\right) = \Psi_{1,n}\Big(\sum_{j,s\geq 0}\binom{s+m-1}{m-1} r_j z^j (tz^k)^s\Big) =$$

$$= \Psi_{1,n}\Big(\sum_{s\geq 0}\binom{s+m-1}{m-1} r_{s(n-k)}\,(tz^n)^s\Big) = \sum_{s\geq 0}\binom{s+m-1}{m-1} r_{s(n-k)} z^s.$$

On other hand

$$\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left(z^{m-1}\varphi_{n-k}(R(z))\right)=\frac{1}{(m-1)!}\left(\sum_{s=0}^{\infty} r_{s(n-k)}z^{m+s-1}\right)_z^{(m-1)}=$$

$$=\frac{1}{(m-1)!}\sum_{s\geq 0}(s+m-1)(s+m-2)\cdots(s+1)r_{s(n-k)}z^s=$$

$$=\sum_{s\geq 0}\binom{s+m-1}{m-1}r_{s(n-k)}z^s.$$

This proves the case $n>k$.

Taking into account the formal property

$$\Psi_{1,n}\left(F(tz^n)\,H(t,z)\right)=F(z)\Psi_{1,n}(H(t,z)),\quad F(z),H(t,n)\in\mathbb{C}[[t,z]],$$

for the case $n=k$ we have

$$\Psi_{1,n}\left(\frac{R(z)}{(1-tz^n)^m}\right)=\frac{1}{(1-z)^m}\Psi_{1,n}\left(R(z)\right)=\frac{R(0)}{(1-z)^m}.$$

To prove the case $n<k$ observe that, the equation $ks+j=ns$ for $n<k$ and $j,s\geq 0$ has only one trivial solution $j=s=0$. We have

$$\Psi_{1,n}\left(\frac{R(z)}{1-tz^k}\right)=\Psi_{1,n}\left(\sum_{j,s\geq 0} r_j z^j(tz^k)^s\right)=\Psi_{1,n}\left(\sum_{j,s\geq 0} r_j t^s z^{ks+j}\right)=r_0=R(0).$$

□

The main idea of the calculations of the paper is that the Poincaré series $\mathcal{PI}_{\mathbf{d}}(z)$, $\mathcal{PS}_{\mathbf{d}}(z)$ can be expressed in terms of functions Ψ. The following simple but important statement holds:

Lemma 3.2. *Let* $d^*:=\max(\mathbf{d})$. *Then*

$$(i)\quad \mathcal{PI}_{\mathbf{d}}(z)=\Psi_{1,d^*}\left((1-z^2)f_{\mathbf{d}}(tz^{d^*},z)\right),$$

$$(ii)\quad \mathcal{PS}_{\mathbf{d}}(z)=\Psi_{1,d^*}\left((1+z)f_{\mathbf{d}}(tz^{d^*},z)\right).$$

P r o o f. Theorem 2.2 states that $\dim(\mathcal{I}_{\mathbf{d}})_n = [t^n](1-z^2)f_{\mathbf{d}}(t,z)$. Then

$$\mathcal{PI}_{\mathbf{d}}(z) = \sum_{n=0}^{\infty} \dim(I_{\mathbf{d}})_n z^n = \sum_{n=0}^{\infty} \big([t^n](1-z^2)f_{\mathbf{d}}(t,z)\big)z^n =$$

$$= \sum_{n=0}^{\infty} \big([(tz^{d^*})^n](1-z^2)f_{\mathbf{d}}(tz^{d^*},z)\big)z^n = \Psi_{1,d^*}\Big((1-z^2)f_{\mathbf{d}}(tz^d,z)\Big).$$

Similarly, we prove the statement (ii).

We replaced t with tz^{d^*} to avoid negative powers of z in the denominator of the function $f_{\mathbf{d}}(t,z)$. □

Write the function $f_{\mathbf{d}}(t,z)$ in the following way

$$f_{\mathbf{d}}(t,z) = \frac{1}{\prod_{k=1}^{s}(tz^{-d_k}, z^2)_{d_k+1}},$$

here $(a,q)_n = (1-a)(1-a\,q)\cdots(1-a\,q^{n-1})$ denotes the q-shifted factorial.

The above lemma implies the following presentations of the Poincaré series via contour integrals:

Lemma 3.3.

$$(i) \quad \mathcal{PI}_{\mathbf{d}}(t) = \frac{1}{2\pi i}\oint_{|z|=1} \frac{1-z^2}{\prod_{k=1}^{s}(tz^{-d_k}, z^2)_{d_k+1}}\frac{dz}{z},$$

$$(ii) \quad \mathcal{PS}_{\mathbf{d}}(t) = \frac{1}{2\pi i}\oint_{|z|=1} \frac{1+z}{\prod_{k=1}^{s}(tz^{-d_k}, z^2)_{d_k+1}}\frac{dz}{z}.$$

P r o o f. We have

$$\mathcal{PS}_{\mathbf{d}}(t) = \sum_{n=0}^{\infty} \dim(I_{\mathbf{d}})_n t^n = \sum_{n=0}^{\infty} \big([t^n](1+z)f_{\mathbf{d}}(t,z)\big)t^n =$$

$$= \sum_{n=0}^{\infty}\left([t^n]\frac{1}{2\pi i}\oint_{|z|=1}(1+z)f_{\mathbf{d}}(t,z)\frac{dz}{z}\right)t^n = \frac{1}{2\pi i}\oint_{|z|=1}(1+z)f_{\mathbf{d}}(t,z)\frac{dz}{z}.$$

Similarly we get the Poincaré series $\mathcal{PI}_{\mathbf{d}}(t)$. □

Note that the Molien-Weyl integral formula for the Poincaré series $\mathcal{P}_d(t)$ of the algebra of invariants of binary d-form can be reduced to the following formula

$$\mathcal{P}_d(t) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{1-z^2}{(1-tz^d)(1-tz^{d-2})\ldots(1-tz^{-d})} \frac{dz}{z} =$$

$$= \frac{1}{2\pi i} \oint_{|z|=1} \frac{1-z^2}{(tz^{-d}, z^2)_{d+1}} \frac{dz}{z}.$$

see [5], p. 183. An ingenious way to calculate such integrals was proposed in [6].

After simplification we can write $f_{\mathbf{d}}(tz^{d^*}, z)$ in the following way

$$f_{\mathbf{d}}(tz^{d^*}, z) = \Big((1-t)^{\beta_0}(1-tz)^{\beta_1}(1-tz^2)^{\beta_2}\ldots(1-tz^{2\,d^*})^{\beta_{2\,d^*}}\Big)^{-1},$$

for some integer $\beta_0, \ldots \beta_{d^*}$. For example

$$f_{(1,2,4)}(tz^4, z) = \frac{1}{(1-t)\,(1-tz^2)^2(1-tz^3)\,(1-tz^4)^2(1-tz^5)\,(1-tz^6)^2(1-tz^8)}.$$

It implies the following partial fraction decomposition of $f_{\mathbf{d}}(tz^{d^*}, z)$:

$$f_{\mathbf{d}}(tz^{d^*}, z) = \sum_{i=0}^{2\,d^*} \sum_{k=1}^{\beta_i} \frac{A_{i,k}(z)}{(1-tz^i)^k},$$

for some polynomials $A_{i,k}(z)$.

By direct calculations we obtain

$$A_{i,k}(z) = \frac{(-1)^{\beta_i - k}}{(\beta_i - k)!\,(z^i)^{\beta_i - k}} \lim_{t \to z^{-i}} \frac{\partial^{\beta_i - k}}{\partial t^{\beta_i - k}} \Big(f_d(tz^{d^*}, z)(1-tz^i)^{\beta_i}\Big).$$

Now we can present Springer type formulas for the Poincaré series $\mathcal{PI}_{\mathbf{d}}(z)$ and $\mathcal{PS}_{\mathbf{d}}(z)$.

Theorem 3.1.

$$\mathcal{PI}_{\mathbf{d}}(z) = \sum_{i=0}^{d^*} \sum_{k=1}^{\beta_i} \frac{1}{(k-1)!} \frac{d^{k-1}\left(z^{k-1}\varphi_{d^*-i}((1-z^2)\,A_{i,k}(z))\right)}{dz^{k-1}},$$

$$\mathcal{PS}_{\mathbf{d}}(z) = \sum_{i=0}^{d^*} \sum_{k=1}^{\beta_i} \frac{1}{(k-1)!} \frac{d^{k-1}\left(z^{k-1}\varphi_{d^*-i}((1+z)\,A_{i,k}(z))\right)}{dz^{k-1}}.$$

P r o o f. Taking into account Lemma 3.1 and the linearity of the map Ψ we get

$$\mathcal{PS}_{\mathbf{d}}(z) = \Psi_{1,d^*}\left((1+z)f_{\mathbf{d}}(tz^{d^*},z)\right) = \Psi_{1,d^*}\left(\sum_{i=0}^{2\,d^*}\sum_{k=1}^{\beta_i}\frac{(1+z)A_{i,k}(z)}{(1-tz^i)^k}\right) =$$

$$= \sum_{i=0}^{d^*}\sum_{k=1}^{\beta_i}\frac{1}{(k-1)!}\frac{d^{k-1}\left(z^{k-1}\varphi_{d^*-i}((1+z)\,A_{i,k}(z))\right)}{dz^{k-1}}.$$

The case $\mathcal{PI}_{\mathbf{d}}(z)$ can be considered similarly. □

Note that the Poincaré series $\mathcal{PI}_d(z)$ and $\mathcal{PC}_d(z)$ of the algebras of invariants and covariants of binary d-form equal

$$\mathcal{PI}_d(z) = \sum_{0\le k<d/2}\varphi_{d-2\,k}\left(\frac{(-1)^k z^{k(k+1)}(1-z^2)}{(z^2,z^2)_k\,(z^2,z^2)_{d-k}}\right),$$

$$\mathcal{PC}_d(z) = \sum_{0\le k<d/2}\varphi_{d-2\,k}\left(\frac{(-1)^k z^{k(k+1)}(1+z)}{(z^2,z^2)_k\,(z^2,z^2)_{d-k}}\right),$$

see [16] and [1] for details.

4. Explicit formulas for small d. The formulas of Theorem 3.1 allow the simplification for some small tuples $\mathbf{d}$.

Theorem 4.1. *Let $s=n$ and $d_1=d_2=\ldots=d_n=1$, i.e. $\mathbf{d}=(1,1,\ldots,1)$. Then*

$$\mathcal{PI}_{\mathbf{d}}(z) = \sum_{k=1}^{n}\frac{(-1)^{n-k}}{(k-1)!}\frac{(n)_{n-k}}{(n-k)!}\frac{d^{k-1}}{dz^{k-1}}\left(\left(\frac{z}{1-z^2}\right)^{2n-k-1}\right),$$

$$\mathcal{PS}_{\mathbf{d}}(z) = \sum_{k=1}^{n}\frac{(-1)^{n-k}}{(k-1)!}\frac{(n)_{n-k}}{(n-k)!}\frac{d^{k-1}}{dz^{k-1}}\left(\frac{(1+z)z^{2n-k-1}}{(1-z^2)^{2n-k}}\right),$$

where $(n)_m := n(n+1)\cdots(n+m-1)$, $(n)_0 := 1$ denotes the shifted factorial.

P r o o f. For $\mathbf{d} = (1, 1, \ldots, 1) \in \mathbb{Z}^n$ we have $d^* = 1$ and

$$f_{\mathbf{d}}(tz^{d^*}, z) = \frac{1}{\big((1-t)(1-tz^2)\big)^n} =$$

$$= \frac{A_{0,1}(z)}{1-t} + \cdots + \frac{A_{0,n}(z)}{(1-t)^n} + R(z), \Psi_{1,1}\left(R(z)\right) = 0,$$

where

$$A_{0,k} = \frac{(-1)^{n-k}}{(n-k)!} \lim_{t \to 1} \frac{\partial^{n-k}}{\partial t^{n-k}} \left(\frac{1}{(1-tz^2)^n} \right).$$

By induction we get

$$\lim_{t \to 1} \frac{\partial^m}{\partial t^m} \left(\frac{1}{(1-tz^2)^n} \right) = (n)_m \frac{(z^2)^m}{(1-z^2)^{n+m}}.$$

Thus,

$$A_{0,k} = \frac{(-1)^{n-k}(n)_{n-k}}{(n-k)!} \frac{(z^2)^{n-k}}{(1-z^2)^{2n-k}}.$$

Now, using Theorem 3.1 and the property $\varphi_1(F(z)) = F(z)$, for any $F(z) \in \mathbb{Z}[[z]]$ we have

$$\mathcal{PS}_{\mathbf{d}}(z) = \Psi_{1,1}\left(\sum_{k=1}^{s} \frac{(1+z)\,A_{0,k}}{(1-t)^k} \right) = \sum_{k=1}^{s} \Psi_{1,1}\left(\frac{(1+z)\,A_{0,k}}{(1-t)^k} \right) =$$

$$= \sum_{k=1}^{n} \frac{1}{(m-1)!} \frac{d^{k-1}}{dz^{k-1}} \left(z^{k-1} \varphi_1((1+z)\,A_{0,k}) \right) =$$

$$= \sum_{k=1}^{n} \frac{1}{(m-1)!} \frac{d^{k-1}}{dz^{k-1}} \left((1+z)\,z^{k-1} A_{0,k} \right) =$$

$$= \sum_{k=1}^{n} \frac{(-1)^{n-k}}{(k-1)!} \frac{(n)_{n-k}}{(n-k)!} \frac{d^{k-1}}{dz^{k-1}} \left(\frac{(1+z) z^{2s-k-1}}{(1-z^2)^{2n-k}} \right).$$

The case $\mathcal{PI}_{\mathbf{d}}(z)$ can be considered similarly. □

Theorem 4.2. *Let $d_1 = d_2 = \ldots = d_n = 2$, $\mathbf{d} = (2, 2, \ldots, 2)$, then*

$$\mathcal{PI}_{\mathbf{d}}(z)=\sum_{k=1}^{n}\frac{(-1)^{n-k}}{(n-k)!(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left(\sum_{i=0}^{n-k}\binom{n-k}{i}\frac{(n)_i(n)_{n-k-i}(1-z)z^{2n-k-i-1}}{(1-z)^{n+i}(1-z^2)^{2n-k-i}}\right),$$

$$\mathcal{PS}_{\mathbf{d}}(z)=\sum_{k=1}^{n}\frac{(-1)^{n-k}}{(n-k)!(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left(\sum_{i=0}^{n-k}\binom{n-k}{i}\frac{(n)_i(n)_{n-k-i}z^{2n-k-i-1}}{(1-z)^{n+i}(1-z^2)^{2n-k-i}}\right).$$

P r o o f. It is easy to check that in this case we have

$$f_{\mathbf{d}}(tz^2, z) = \frac{1}{\big((1-t)(1-tz^2)(1-tz^4)\big)^n}.$$

The decomposition $f_{\mathbf{d}}(tz^2, z)$ into partial fractions yields

$$f_{\mathbf{d}}(tz^2, z) = \sum_{k=1}^{n}\left(\frac{A_k(z)}{(1-t)^k} + \frac{B_k(z)}{(1-tz^2)^k} + \frac{C_k(z)}{(1-tz^4)^k}\right),$$

for some rational functions $A_k(z), B_k(z), C_k(z)$. Then

$$\mathcal{PS}_{\mathbf{d}}(z) = \Psi_{1,2}\left((1+z)f_{\mathbf{d}}(tz^2, z)\right) =$$

$$= \sum_{k=1}^{n}\left(\Psi_{1,2}\left(\frac{(1+z)A_k(z)}{(1-t)^k}\right) + \Psi_{1,2}\left(\frac{(1+z)B_k(z)}{(1-tz^2)^k}\right) + \Psi_{1,2}\left(\frac{(1+z)C_k(z)}{(1-tz^4)^k}\right)\right).$$

Lemma 3.1 implies that

$$\Psi_{1,2}\left(\frac{(1+z)C_k(z)}{(1-tz^4)^k}\right) = 0,$$

and

$$\Psi_{1,2}\left(\frac{(1+z)B_k(z)}{(1-tz^2)^k}\right) = \frac{B_k(0)}{(1-z)^k}, k = 1, \ldots, n.$$

But

$$B_k(z) = \frac{(-1)^{n-k}}{(n-k)!(z^2)^{n-k}}\lim_{t\to z^{-2}}\frac{\partial^{n-k}}{\partial t^{n-k}}\left(\frac{1}{(1-t)^n(1-tz^4)^n}\right).$$

It is easy to see that this partial derivative has the following form

$$\frac{\partial^{n-k}}{\partial t^{n-k}}\left(\frac{1}{(1-t)^n(1-tz^4)^n}\right) = \frac{\overline{B}_k(t,z)}{((1-t)(1-tz^4))^{2n-k}},$$

for some polynomial $\overline{B}_k(t,z)$. Moreover, $\overset{\circ}{t}(\overline{B}_k(t,z)) = n-k$. Then

$$B_k(z) = \frac{(-1)^{n-k}}{(n-k)!(z^2)^{n-k}} \lim_{t\to z^{-2}} \frac{\overline{B}_k(t,z)}{((1-t)(1-tz^4))^{2n-k}} =$$

$$= \frac{(-1)^{n-k} z^{2n} \overline{B}_k(1/z^2,z)}{(n-k)!((z^2-1)(1-tz^4))^{2n-k}}.$$

It follows that $B_k(z)$ has the factor z^{2k} and then $B_k(0)=0$. Thus

$$\Psi_{1,2}\left(\frac{(1+z)B_k(z)}{(1-tz^2)^k}\right) = 0, k=1,\ldots,n.$$

Therefore

$$\mathcal{PS}_{\mathbf{d}}(z) = \sum_{k=1}^{n} \Psi_{1,2}\left(\frac{(1+z)A_k(z)}{(1-t)^k}\right) =$$

$$= \sum_{k=1}^{n} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left(z^{k-1}\varphi_2((1+z)A_k(z))\right).$$

Let us to calculate $A_k(z)$. We have

$$A_k(z) = \frac{(-1)^{n-k}}{(n-k)!} \lim_{t\to 1} \frac{d^{n-k}}{dt^{n-k}} \left(f_{\mathbf{d}}(tz^2,z)(1-t)^n\right) =$$

$$= \frac{(-1)^{n-k}}{(n-k)!} \lim_{t\to 1} \frac{d^{n-k}}{dt^{n-k}} \left(\frac{1}{(1-tz^2)^n(1-tz^4)^n}\right) =$$

$$= \frac{(-1)^{n-k}}{(n-k)!} \lim_{t\to 1} \sum_{i=0}^{n-k} \binom{n-k}{i} \left(\frac{1}{(1-tz^2)^n}\right)_t^{(i)} \left(\frac{1}{(1-tz^4)^n}\right)_t^{(n-k-i)} =$$

$$= \frac{(-1)^{n-k}}{(n-k)!} \lim_{t\to 1} \sum_{i=0}^{n-k} \binom{n-k}{i} (n)_i (n)_{n-k-i} \frac{z^{2i}}{(1-tz^2)^{n+i}} \frac{z^{4(n-k-i)}}{(1-tz^4)^{2n-k-i}} =$$

$$= \frac{(-1)^{n-k}}{(n-k)!} \sum_{i=0}^{n-k} \binom{n-k}{i} (n)_i (n)_{n-k-i} \frac{(z^2)^{2(n-k)-i}}{(1-z^2)^{n+i}(1-z^4)^{2n-k-i}}.$$

Taking into account that $\varphi_2(F(z^2)) = F(z)$, and $\varphi_2(zF(z^2)) = 0$ we obtain

$$\varphi_2((1+z)A_k(z)) = \varphi_2(A_k(z)) =$$
$$= \frac{(-1)^{n-k}}{(n-k)!}\sum_{i=0}^{n-k}\binom{n-k}{i}(n)_i(n)_{n-k-i}\frac{(z)^{2(n-k)-i}}{(1-z)^{n+i}(1-z^2)^{2n-k-i}}.$$

Thus,

$$\mathcal{PS}_{\mathbf{d}}(z) = \sum_{k=1}^{n}\Psi_{1,2}\left(\frac{(1+z)A_k(z)}{(1-t)^k}\right) =$$
$$= \sum_{k=1}^{n}\frac{1}{(k-1)!}\frac{d^{k-1}}{dt^{k-1}}\left(z^{k-1}\varphi_2((1+z)A_k(z)\right) =$$
$$= \sum_{k=1}^{n}\frac{(-1)^{n-k}}{(n-k)!(k-1)!}\frac{d^{k-1}}{dz^{k-1}}\left(\sum_{i=0}^{n-k}\binom{n-k}{i}\frac{(n)_i(n)_{n-k-i}z^{2n-k-i-1}}{(1-z)^{n+i}(1-z^2)^{2n-k-i}}\right).$$

By replacing the factor $1+z$ with $1-z^2$ in $\mathcal{PS}_{\mathbf{d}}(z)$ and taking into account that
$$\varphi_2((1-z^2)A_k(z)) = (1-z)\varphi_2(A_k(z)),$$
we get the Poincaré series $\mathcal{PI}_{\mathbf{d}}(z)$. □

5. Examples. For direct computations of the function φ_n we use the following technical lemma, see [1]:

Lemma 5.1. *Let $R(z)$ be polynomial of z. Then*

$$\varphi_n\left(\frac{R(z)}{(1-z^{k_1})(1-z^{k_2})\cdots(1-z^{k_m})}\right) = \frac{\varphi_n\big(R(z)Q_n(z^{k_1})Q_n(z^{k_2})Q_n(z^{k_m})\big)}{(1-z^{k_1})(1-z^{k_2})\cdots(1-z^{k_m})},$$

here $Q_n(z) = 1+z+z^2+\cdots+z^{n-1}$, and k_i are natural numbers.

As example, let us calculate the Poincaré series $\mathcal{PD}_{(1,2,3)}$. We have $d^* = 3$ and

$$f_{(1,2,3)}(t,z) = \frac{1}{(1-tz^4)^2\,(1-tz^2)^2\,(1-tz^5)\,(1-tz^3)\,(1-tz)\,(1-tz^6)\,(1-t)}.$$

The decomposition $f_{(1,2,3)}(t,z)$ into partial fractions yields:

$$f_{(1,2,3)}(t,z)=\frac{A_{0,1}(z)}{1-t}+\frac{A_{1,1}(z)}{1-tz}+\frac{A_{2,1}(z)}{1-tz^2}+\frac{A_{2,2}(z)}{(1-tz^2)^2}+\frac{A_{3,1}(z)}{1-tz^3}+\frac{A_{4,1}(z)}{1-tz^4}+$$

$$+\frac{A_{4,2}(z)}{(1-tz^4)^2}+\frac{A_{5,1}(z)}{1-tz^5}+\frac{A_{6,1}(z)}{1-tz^6}.$$

By using Lemma 3.1 we have

$$\mathcal{PD}_{(1,2,3)}(z)=\Psi_{1,3}\left((1+z)f_{(1,2,3)}(t,z)\right)=$$

$$=\Psi_{1,3}\left(\frac{(1+z)A_{0,1}(z)}{1-t}\right)+\Psi_{1,3}\left(\frac{(1+z)A_{1,1}(z)}{1-tz}\right)+$$

$$+\Psi_{1,3}\left(\frac{(1+z)A_{2,1}}{1-tz^2}\right)+\Psi_{1,3}\left(\frac{(1+z)A_{2,2}(z)}{(1-tz^2)^2}\right)+\Psi_{1,3}\left(\frac{(1+z)A_{3,1}(z)}{1-tz^3}\right)=$$

$$=\varphi_3\left((1+z)A_{0,1}(z)\right)+\varphi_2\left((1+z)A_{1,1}(z)\right)+\varphi_1\left((1+z)A_{2,1}(z)\right)+$$

$$+\left(z\varphi_1\left((1+z)A_{2,2}(z)\right)\right)'_z+A_{3,1}(0).$$

Now

$$A_{0,1}(z)=\lim_{t\to 1}\left(f_{(1,2,3)}(t,z)(1-t)\right)=$$

$$=\frac{1}{(1-z^4)^2(1-z^2)^2(1-z^5)(1-z^3)(1-z)(1-z^6)}.$$

and

$$\varphi_3((1+z)A_{0,1}(z))=$$

$$=\frac{2z^{11}+7z^{10}+14z^9+29z^8+34z^7+42z^6+42z^5+33z^4+21z^3+14z^2+4z+1}{(1-z^5)(1-z)^3(1-z^4)^2(1-z^2)^2}.$$

As above we obtain

$$A_{1,1}(z)=\lim_{t\to z^{-1}}\left(f_{(1,2,3)}(t,z)(1-tz)\right)=$$

$$=\frac{z}{(1-z^3)^2(1-z)^2(1-z^4)(1-z^2)(1-z^5)(z-1)},$$

$$\varphi_2((1+z)A_{1,1}(z)) = -\frac{z\left(4+13\,z^2+6\,z+6\,z^6+z^7+13\,z^4+9\,z^5+12\,z^3\right)}{(1-z^2)\,(1-z^5)\,(1-z^3)^2\,(1-z)^4}.$$

$$A_{2,1}(z) = -\frac{1}{z^2}\lim_{t\to z^{-2}}\left(f_{(1,2,3)}(t,z)(1-tz^2)^2\right)'_t =$$

$$= -\frac{z^3\left(5\,z^6+5\,z^5+6\,z^4+2\,z^3-z^2-2\,z-2\right)}{(1-z^4)^2\,(1-z)\,(1-z^3)^2\,(1-z^2)^3},$$

$$\varphi_1((1+z)A_{2,1}(z)) = -\frac{z^3\left(5\,z^6+5\,z^5+6\,z^4+2\,z^3-z^2-2\,z-2\right)}{(1-z^4)^2\,(1-z)^2\,(1-z^3)^2\,(1-z^2)^2}.$$

$$A_{2,2}(z) = \lim_{t\to z^{-2}}\left(f_{(1,2,3)}(t,z)(1-tz)^2\right) = \frac{z^3}{(1-z^4)\,(1-z)^3\,(1-z^3)\,(1-z^2)^2},$$

$$(z\varphi_1\left((1+z)A_{2,2}(z)\right))'_z = (z(1+z)A_{2,2}(z))'_z =$$

$$= \frac{z^3\left(10\,z^6+13\,z^5+20\,z^4+16\,z^3+14\,z^2+7\,z+4\right)}{(1-z^2)^2\,(1-z^3)^2\,(1-z)^2\,(1-z^4)^2}.$$

At last

$$A_{3,1}(z) = \lim_{t\to z^{-3}}\left(f_{(1,2,3)}(t,z)(1-tz^3)\right) = \frac{z^7}{(1-z^3)^2\,(1-z)^5\,(1-z^2)}.$$

Thus $A_{3,1}(0)=0$.

After summation and simplification we obtain the explicit expression for the Poincaré series

$$\mathcal{PD}_{(1,2,3)}(z) = \frac{p_{(1,2,3)}(z)}{(1-z^4)^2\,(1-z)^2\,(1-z^2)\,(1-z^3)^2\,(1-z^5)},$$

where

$$p_{1,2,3}(z) = z^{14}+z^{13}+6z^{12}+12z^{11}+20z^{10}+29z^9+35z^8+39z^7+35z^6+$$

$$+29z^5+20\,z^4+12z^3+6z^2+z+1.$$

The following Poincaré series are obtained using the explicit formulas of Theorem 3.2 and Theorem 3.3

$$\mathcal{PD}_{(1,1)}(z) = \frac{1}{(1-z)^2\,(1-z^2)},\quad \mathcal{PD}_{(1,1,1)}(z) = \frac{1-z^3}{(1-z)^3\,(1-z^2)^3},$$

$$\mathcal{PD}_{(1,1,1,1)}(z) = \frac{z^4+2z^3+4z^2+2z+1}{(1-z)^2\,(1-z^2)^5},$$

$$\mathcal{PD}_{(1,1,1,1,1)}(z) = \frac{z^6+3z^5+9z^4+9z^3+9z^2+3z+1}{(1-z)^2\,(1-z^2)^7},$$

$$\mathcal{PD}_{(1,1,1,1,1,1)}(z) = \frac{z^8+4z^7+16z^6+24z^5+36z^4+24z^3+16z^2+4z+1}{(1-z)^2\,(1-z^2)^9}$$

$$\mathcal{PD}_{(1,1,1,1,1,1,1)}(z) = \frac{p_7(z)}{(1-z)^2\,(1-z^2)^{11}}$$

$$p_7(z) = z^{10}+5z^9+25z^8+50z^7+100z^6+100z^5+100z^4+50z^3+25z^2+5z+1.$$

$$\mathcal{PD}_{(2,2,2)}(z) = \frac{1+4z^2+z^4}{(1-z)^3\,(1-z^2)^5}, \quad \mathcal{PD}_{(2,2,2,2)}(z) = \frac{1+9z^2+9z^4+z^6}{(1-z)^4\,(1-z^2)^7}$$

$$\mathcal{PD}_{(2,2,2,2,2)}(z) = \frac{1+16z^2+36z^4+16z^6+z^8}{(1-z)^5\,(1-z^2)^9},$$

$$\mathcal{PD}_{(2,2,2,2,2,2)}(z) = \frac{z^{10}+25z^8+100z^6+100z^4+25z^2+1}{(1-z)^6\,(1-z^2)^{11}},$$

$$\mathcal{PD}_{(2,2,2,2,2,2,2)}(z) = \frac{z^{12}+36z^{10}+225z^8+400z^6+225z^4+36z^2+1}{(z-1)^7\,(1-z^2)^{13}}.$$

By using Maple we computed the Poincaré series up to $n = 30$. The cases $n = 2, 3$ agree with the results of the papers [2], [7].

REFERENCES

[1] L. Bedratyuk. The Poincaré series of the covariants of binary form. arXiv:0904.1325

[2] L. Bedratyuk. The Poincaré series of the algebras of simultaneous invariants and covariants of two binary forms. *Linear and Multilinear Algebra*, to appear.

[3] A. Brouwer. The Poincaré series, `http://www.win.tue.nl/~aeb/math/poincare.html`

[4] A. Brouwer, A.Cohen. The Poincare series of the polynomial invariants under SU_2 in its irreducible representation of degree ≤ 17. Preprint of the Mathematisch Centrum, Amsterdam, 1979.

[5] H. Dersken, G. Kemper. Computational Invariant Theory. Springer-Verlag, New York, 2002.

[6] D. Doković. A heuristic algorithm for computing the Poincare series of the invariants of binary forms. *Int. J. Contemp. Math. Sci.* **1** (2006), 557–568.

[7] V. Drensky, G. K. Genov. Multiplicities of Schur functions with applications to invariant theory and PI-algebras. *[J] C. R. Acad. Bulg. Sci.* **57,** 3 (2004) 5–10.

[8] W. Fulton, J. Harris. Representation theory. A first course, Graduate Texts in Mathematics, **129**, New York etc., Springer-Verlag, 1991, xv+551 p.

[9] J. Grace, A. Young. The Algebra of Invariants. Cambrige Univ. Press, 1903.

[10] F. Grosshans. Observable groups and Hilbert's fourteenth problem. *Amer. J. Math.* **95** (1973), 229–253.

[11] D. Hilbert. Theory of algebraic invariants, Cambridge University Press, 1993.

[12] P. Littelman, C.Procesi. On the Poincaré series of the invariants of binary forms. *J. Algebra* **133**, *2* (1990), 490–499.

[13] N. Onoda. Linear action of G_a on polynomial rings. Proc. 25th Symp. Ring Theory, Matsumoto, 1992, 11–16.

[14] K. Pommerening. Invariants of unipotent groups – A survey. In: Invariant theory, Symp. West Chester/Pa. 1985, Lect. Notes Math. vol. **1278**, 1987, 8–17.

[15] C. S. Seshadri. On a theorem of Weitzenböck in invariant theory. *J. Math. Kyoto Univ.* **1** (1962), 403–409.

[16] T. A. Springer. On the invariant theory of SU(2). *Indag. Math.* **42** (1980), 339–345.

[17] J. J. Sylvester, F. Franklin. Tables of the generating functions and groundforms for the binary quantic of the first ten orders. *Am. J. II.,* (1879) 223–251.

[18] J. J. Sylvester. Tables of the generating functions and groundforms of the binary duodecimic, with some general remarks, and tables of the irreducible syzygies of certain quantics. *Am. J. IV.* (1881)41–62.

[19] A. Tyc. An elementary proof of the Weitzenböck theorem. *Colloq. Math.* **78** (1998), 123–132.

[20] R. Weitzenböck. Über die Invarianten von linearen Gruppen. *Acta Math.* **58** (1932), 231–293.

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